Published by Institute of Physics Publishing for SISSA
Received: *July 18, 2006*
Accepted: *September 26, 2006*
Published: *October 18, 2006*

Can D-branes wrap nonrepresentable cycles?

Jarah Evslin[a] and Hisham SATI[bcd]

[a] *International Solvay Institutes, Physique Théorique et Mathématique*
Université Libre de Bruxelles
C.P. 231, B-1050, Bruxelles, Belgium
[b] *Department of Mathematics, Yale University*
New Haven, CT 06520, U.S.A.
[c] *Department of Pure Mathematics, University of Adelaide*
Adelaide, SA 5005, Australia
[d] *The Erwin Schrödinger International, Institute for Mathematical Physics,*
Boltzmanngasse 9, A-1090 Wien, Austria
E-mail: jevslin@ulb.ac.be, hisham.sati@adelaide.edu.au

Abstract: Sometimes a homology cycle of a nonsingular compactification manifold cannot be represented by a nonsingular submanifold. We want to know whether such nonrepresentable cycles can be wrapped by D-branes. A brane wrapping a representable cycle carries a K-theory charge if and only if its Freed-Witten anomaly vanishes. However some K-theory charges are only carried by branes that wrap nonrepresentable cycles. We provide two examples of Freed-Witten anomaly-free D6-branes wrapping nonrepresentable cycles in the presence of a trivial NS 3-form flux. The first occurs in type IIA string theory compactified on the Sp(2) group manifold and the second in IIA on a product of lens spaces. We find that the first D6-brane carries a K-theory charge while the second does not.

Keywords: D-branes, Differential and Algebraic Geometry.

JHEP10(2006)050

Contents

1. Introduction

D-branes are not classified by homology. For example in type II string theory on a *spin* spacetime with a topologically trivial NSNS 3-form flux, Freed and Witten [1] have shown that D-branes can only wrap $spin^c$ submanifolds. Maldacena, Moore and Seiberg (MMS) [2] demonstrated that even in the SU(3) WZW model some homology classes of the spacetime are not represented by $spin^c$ manifolds and so cannot be wrapped by D-branes. With such inconsistent D-branes removed from the spectra, MMS demonstrated that the conserved brane charges in this example are classified not by homology but by twisted K-theory, in line with the conjectures of Refs. [3 – 5].

In general the Freed-Witten (FW) anomaly is a necessary but not a sufficient condition for the homology class of a D-brane to lift to a twisted K-theory class. We will argue that, when the NS 3-form H is topologically trivial, the FW anomaly is a necessary and sufficient condition for all Dp-branes except for D6-branes. D6-branes are special because they wrap 7-dimensional cycles in a 10-dimensional spacetime. René Thom, in work that won him the 1958 Fields Medal, demonstrated that this is the lowest-dimensional case in which a homology cycle may not be representable by a nonsingular submanifold. This leads us to the question: *Can D-branes wrap nonrepresentable cycles?* To answer this question definitively one should look at the worldsheet theory of fundamental strings, impose boundary conditions corresponding to a singular representative of the cycle and check for inconsistencies, such as a failure of BRST invariance. In the present note we will use a less reliable method. We will check to see whether branes wrapping nonrepresentable cycles carry K-theory charges. We will find that the answer is *yes* for some cycles and *no* for others.

Although one can show that FW anomaly-free D6-branes that fail to carry a K-theory charge always wrap nonrepresentable cycles, the converse does not always hold. Some K-theory charges are only carried by nonrepresentable branes. To demonstrate this, we will recall an example of a D6-brane on a nonrepresentable cycle from Ref. [6] and show that it does carry a K-theory charge[1]. Thus the type IIA version [8] of the Sen conjecture [9] implies that D-branes may wrap certain nonrepresentable cycles.

An example[2] of a FW anomaly-free D6-brane that does not carry a K-theory charge was presented in Refs. [10, 11]. We will argue that the wrapped 7-cycle is not represented by any nonsingular submanifold, which in particular implies that its singularity cannot be removed by deformations or blowups. We find that the singular locus is homologous to a two-torus.

In Sec. 2 we explain the relation between homology and K-theory charges. We review an algorithm for calculating K-theory from homology, and in particular we find the obstructions to lifting a homology class to a K-theory class. In Sec. 3 we use a result of René Thom to argue that all of these obstructions are summarized by the Freed-Witten anomaly together with a necessary but not sufficient condition for the representability of the wrapped homology cycle by a nonsingular submanifold. Then in Sec. 4 we will present two examples of the second kind of obstruction, a D6-brane wrapping a nonrepresentable 7-cycle in the group manifold Sp(2) and in a product of lens spaces. We end with some discussion in Sec. 5.

2. The Atiyah-Hirzebruch Spectral Sequence

A D-brane that wraps a nontrivial cycle carries a charge that corresponds to the homology class of the cycle. Diaconescu, Moore and Witten (DMW) [12] have shown that not all of these charges are conserved, instead there are dynamical processes in which branes wrapping nontrivial cycles can decay. In addition, in Ref. [1] the authors have found that certain cycles cannot be wrapped by single branes. They argued that any brane wrapping such a cycle would be anomalous, and in fact evidence was presented in [12] that their contributions to the partition function cancel. Thus to compute the partition function it suffices to restrict one's attention to equivalence classes of anomaly-free branes. In other words, D-branes are classified by a quotient of a subset of homology.

MMS have argued that this quotient of a subset is precisely twisted K-theory. They used a mathematical algorithm known as the Atiyah-Hirzebruch spectral sequence (AHSS) to determine which homology classes lift to K-theory classes, that is, to determine which D-branes are unstable and which are not allowed. While in their examples the anomalous branes suffered from the Freed-Witten (FW) anomaly, in general the AHSS construction eliminates some branes that are FW anomaly-free. This leads to the question of whether

[1]The generalized D-branes of Ref. [7] may, implicitly, wrap nonrepresentable cycles.

[2]We have been informed that the first demonstration of the nonrepresentability of this cycle is in version one of Ref. [6], which is also available on the arXiv at the same URL as the current version. However this reference does not address the issue of whether a brane wrapping this cycle carries a K-theory charge, which is the focus of the current note.

the branes that are eliminated by the AHSS construction, but not by the FW anomaly, are allowed in the physical theory. If such branes are allowed, they would provide counterexamples to the K-theory classification program and to the IIA version of the Sen conjecture. On the other hand, if such branes are not allowed, they would be examples of a new anomaly. In the present note we will adopt the more modest goal of providing a characterization of these branes.

The AHSS consists of a series of differential operators d_{2p-1}, $p \geq 2$ which map elements of the qth integral cohomology to elements of the $(q+2p-1)$th cohomology

$$d_{2p-1} : \mathrm{H}^q \longrightarrow \mathrm{H}^{q+2p-1}, \qquad p = 2, 3, \cdots . \tag{2.1}$$

In general the differential d_{2p-1} consists of cohomology operations on the free parts of the integral cohomology and also on cyclic subgroups of prime order less than or equal to p. In particular, in the case of untwisted K-theory and when p is prime, d_{2p-1} contains a primary cohomology operation known as the first Milnor primitive [3] Q_1 whose image is a p torsion class $\mathbb{Z}_p$, and also it may contain secondary operations whose images are torsion at lower primes. A secondary cohomology operation is an operation that is not defined on the entire cohomology, but is defined on the kernels of the preceding differentials.

We will use Poincaré duality to identify a D$(9-q)$-brane wrapping a $(10-q)$-cycle in the integral homology group H_{10-q} with its dual cocycle in H^q, which in terms of supergravity fields corresponds to the Ramond-Ramond source dG_{q-1}. The homology class of a D-brane wrapping the cycle N_{10-q} lifts to a twisted K-theory class if and only if its dual cohomology class $\mathrm{PD}(N_{10-q})$ is in the kernel of all of the differentials

$$d_{2p-1}(\mathrm{PD}(N_{10-q})) = 0 \quad \text{for all } p. \tag{2.2}$$

For example, the first nontrivial differential contains a primary operation at prime 2 and can be explicitly written

$$d_3 x = Q_1 x + H \cup x = Sq^3 x + H \cup x \tag{2.3}$$

where H is the NSNS 3-class and $\cup$ is the cup product, the integral version of the wedge product. The Milnor primitive Q_1 at prime 2 is often denoted Sq^3 and is called a Steenrod square or more precisely square 3. Sq^3, like the cup product with H, increases the degree of a cohomology class by three. DMW have explained that if d_3 does not annihilate the class of a brane then the brane suffers from an FW anomaly. The converse is not true since some FW anomalous branes are annihilated by d_3. MMS have found an example of this phenomenon in the SU(3) Wess-Zumino-Witten WZW model, and in that case the offending class was not in the kernel of d_5 and so, as expected, did not lift to twisted K-theory.

More concretely, consider a brane with worldvolume N in the spacetime M. Let $i : N \hookrightarrow M$ be the inclusion map of the brane into the spacetime. Then the FW anomaly is [1]

$$W_3 + H = 0 \tag{2.4}$$

[3] Higher Milnor primitives appear in the differentials of 'higher' generalized cohomology theories, for example Q_2 appears in Morava K-theory and elliptic cohomology [13].

JHEP10(2006)050

where W_3 is the third integral Stiefel-Whitney class of the normal bundle of N in M and H is the pullback of the NSNS 3-form to the brane worldvolume N. The pushforward of $W_3 + H$ to the spacetime M is

$$i_*(W_3 + H) = Sq^3(\mathrm{PD}(N)) + H \cup \mathrm{PD}(N) = d_3(\mathrm{PD}(N)). \tag{2.5}$$

In the aforementioned SU(3) example $W_3 + H$ is nontrivial but it is in the kernel of the pushforward. This example suggests that the role of the secondary operations is to pick up the anomalies that were in the kernel of the pushforward. In particular one may conjecture that the secondary operations do not imply the existence of any new anomalies, for example all of the two-torsion operations encode the FW anomaly.

3. Nonrepresentable Cycles

While the mod 2 Milnor primitive $Q_1 = -Sq^3$ captures the FW anomaly, the mod 3 Milnor primitive Q_1 is insensitive to 2-torsion characteristic classes like the third Stiefel-Whitney class and so it describes a different anomaly. Our next goal will be to characterize the worldvolumes of the branes suffering from this new anomaly. We will restrict our attention to the case in which the NS H flux is topologically trivial. By this we mean not only that H is exact as a differential form but further that it represents the trivial class of the full integral cohomology.

Imagine that the cycle N wrapped by our D-brane is a nonsingular manifold with a *spin*^{c} normal bundle, embedded as usual in a *spin* spacetime. Consider a trivial, rank one vector bundle on our D-brane. This defines a nontrivial class in the K-theory of N. As the normal bundle to N is *spin*^{c}, we may push this class forward into the K-theory with compact support on a tubular neighborhood of N in M. We may then push this class forward yet again into the K-theory of M without obstruction, and we will obtain the (possibly trivial) K-class which is the K-theory lift of the cohomology Poincaré dual of N. Thus the cohomology Poincaré dual of N lifts to a K-class whenever N is a nonsingular manifold with a *spin*^{c} normal bundle. The *spin*^{c} normal bundle condition is physically just the condition that the brane not have a Freed-Witten anomaly. The AHSS indicates that there must be other obstructions arising at other primes, but the existence of the above pushforward construction of the K-class suggests that if N is nonsingular then there is no other obstruction. Thus the Q_1's at higher primes, which appear in the higher differentials, may only be obstructions to the existence of a nonsingular manifold representing the homology class of N.

This has been known in the mathematics literature for more than half a century. René Thom proved [14] that any cohomology operation at an odd prime, at any dimension not equal to zero modulo four, annihilates all cohomology classes which are dual to homology classes that are representable by nonsingular manifolds. The primes greater than two are all odd and the Q_1's are all of odd degree, which are not equal to zero modulo four. Therefore Thom's theorem implies that Q_1 at every prime greater than two annihilates all cohomology classes dual to cycles that can be represented by nonsingular submanifolds.

JHEP10(2006)050

In particular, up to secondary operations, a D-brane which does not lift to a K-theory class is not in the kernel of Q_1 at some prime. If the D-brane does not have an FW anomaly, then it is in the kernel of Q_1 at prime 2, and so it must not be in the kernel of Q_1 at some odd prime. Therefore its homology class is not representable by a nonsingular manifold.

In critical superstring theories the spacetime is a 10-dimensional manifold. The lowest dimensional Q_1 that measures an obstruction to the representability of a homology class occurs at prime 3. It is $Q_1 = -\beta P_3^1$, where β is the Bockstein homomorphism which raises the cohomology degree by one,

$$\beta \ : \ H^{2j+1}(X, \mathbb{Z}_3) \longrightarrow H^{2j+2}(X, \mathbb{Z}_3), \tag{3.1}$$

arising from the exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow \mathbb{Z}_9 \longrightarrow \mathbb{Z}_3 \longrightarrow 0. \tag{3.2}$$

P_3^1 is the first Steenrod power operation at the prime 3

$$P_3^1 \ : \ H^k(X, \mathbb{Z}_3) \longrightarrow H^{k+2(3-1)}(X, \mathbb{Z}_3). \tag{3.3}$$

In particular Q_1 annihilates any cocycle of degree less than 3, and so its image is a 3-torsion cocycle of degree equal to at least 8. A 10-manifold cannot have 3-torsion at degree 10, as the degree 10 cohomology is determined entirely by the manifold's orientability, which is in $\mathbb{Z}_2$, and so Q_1 annihilates 5-classes. It turns out that Q_1 also annihilates 4-classes. Therefore Q_1 may only be nontrivial on a 3-class z. Physically z may describe the Ramond-Ramond 3-form flux G_3 in type IIB or a D6-brane in type IIA which is Poincaré dual to z. In the first case $\beta P_3^1 z$ is the D-string charge carried by the flux. In the second the dual of $\beta P_3^1 z$ is the singular locus of the D6. In M-theory, a role for $p = 3$ has appeared in [15].

4. Two Examples

4.1 IIA on a Product of Lens Spaces

We now recall an example of a nontrivial Q_1 action that has appeared in Refs. [6, 10, 11]. Consider the product of lens spaces $X^{10} = S^3/\mathbb{Z}_3 \times S^7/\mathbb{Z}_3$, where the $\mathbb{Z}_3$'s are subgroups of the free circle actions on the spheres. T-duality and fluxes on similar spaces has been considered in [16]. We will be interested in the cohomology groups of $S^3/\mathbb{Z}_3$ and $S^7/\mathbb{Z}_3$ with $\mathbb{Z}_3$ coefficients. These are generated by the 1 and 2-cocycles x_1 and x_2 for $S^3/\mathbb{Z}_3$, and y_1 and y_2 for $S^7/\mathbb{Z}_3$.

The cocycles x_1 and y_1 do not have integer lifts, as $\beta(x_1) = -x_2$ and $\beta(y_1) = -y_2$. However the degree three class

$$w = x_1 y_2 - x_2 y_1 \tag{4.1}$$

does admit an integer lift as

$$\beta(w) = -x_2 y_2 + x_2 y_2 = 0. \tag{4.2}$$

We are interested in the action of $Q_1 = -\beta P_3^1$ on w, which gives the non-zero result

$$d_5(w) = Q_1(w) = -\beta P_3^1(w) = -\beta P_3^1(x_1 y_2) = -\beta(x_1 y_2^3) = x_2 y_2^3. \tag{4.3}$$

Thus the 7-cycle Poincaré dual to w is not representable by any nonsingular submanifold, and, as it is not annihilated by the AHSS differential d_5, it also does not lift to a K-theory class. The obstruction is $d_5(w)$ which is dual, inside of the 7-cycle, to a 2-torus. This suggests that, at least for a certain choice of representatives of the 7-cycle, the singular locus is a 2-torus. For example a 5-dimensional normal slice to the 2-torus inside of the singular 7-cycle may be the real cone over $\mathbb{CP}^2$.

4.2 IIA on the Sp(2) Group Manifold

We have used the AHSS and the critical dimension of type II superstring theories to argue that an FW anomaly-free brane lifts to twisted K-theory if and only if the Poincaré dual w of the cycle that it wraps is in the kernel of $Q_1 = -\beta P_3^1$. Thom's result on representability is somewhat weaker, while all representable cycles are in the kernel of Q_1, there are other obstructions to representability that do not appear in the AHSS. For example, in Ref. [6] the authors have shown that representability of the dual cycle to w implies

$$w \cup P_3^1 w = 0. \tag{4.4}$$

In particular they provided an example in which $\beta P_3^1 w = 0$ and $w \cup P_3^1 w \neq 0$. They considered the 10-dimensional group manifold Sp(2). This is topologically a 3-sphere bundle over a 7-sphere and in particular it has the cohomology ring of the trivial bundle

$$\mathrm{H}^0(\mathrm{Sp}(2)) = \mathrm{H}^3(\mathrm{Sp}(2)) = \mathrm{H}^7(\mathrm{Sp}(2)) = \mathrm{H}^{10}(\mathrm{Sp}(2)) = \mathbb{Z}. \tag{4.5}$$

Notice that the cohomology contains no torsion subgroups and so all of the Atiyah-Hirzebruch differentials are trivial and every integral cohomology class lifts to a K-theory class. However the authors prove that the generator w of $\mathrm{H}^3 = \mathbb{Z}$ does not satisfy the condition (4.4) and so the dual homology class is not representable. Thus in this case there is a K-class which does not correspond to any homology class which is represented by any nonsingular submanifold. As every K-class is realizable as a gauge field configuration on some stack of branes, the IIA version of the Sen conjecture implies that such a singular brane configuration must be allowed in IIA string theory.

One may then hope to use the Sp(2) WZW model to provide a nontrivial test of the Sen conjecture. However notice that in the example at hand we have considered a trivial H-flux, corresponding to a negative level and therefore a nonunitary conformal field theory. It may therefore be difficult to decide whether this brane configuration should be allowed.

5. Discussion

We have argued that an FW anomaly-free brane carries a K-theory charge if it wraps a representable cycle. We then presented two examples that showed that the nonrepresentability of a FW anomaly-free cycle does not mean that a wrapped brane necessarily does or does not carry a K-theory charge.

In particular we considered one compactification on a product of lens spaces in which a FW anomaly-free brane wrapped on a nonrepresentable cycle does not carry a K-theory charge. One may object that this is not a legitimate compactification of IIA string theory as it has positive curvature and so is not Ricci flat. This problem is easily solved. The product of lens spaces is a T^2-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^3$ and can be made into a Ricci-flat space by replacing $\mathbb{CP}^1$ by a Riemann surface of genus greater than zero. As this replacement does not affect the calculation of Q_1 [10], there will still be a nonrealizable cycle which does not lift to a K-theory class.

One may argue that branes on nonrepresentable cycles are of limited interest, because in string phenomenology one is typically interested in a four-dimensional topologically trivial space times a six, seven or eight-manifold, and all cycles of six, seven and eight-manifolds are representable. There are two interesting cases in which representability is an issue. First, one may consider noncritical string theories. For example nonrepresentable cycles are generic on group manifolds and so in WZW models. Second, one may consider a spacetime which locally is a product of $\mathbb{R}^4$ or dS^4 times a low-dimensional manifold, but in which globally the topology of the four big dimensions is mixed with that of the little dimensions. For example, at the big bang, at a big crunch and at some horizons the product approximation may fail. Such a compactification would break four-dimensional Lorentz symmetry a little far away from interesting places like the big bang, and break it a lot at the big bang. Of course, this is the breaking pattern observed in nature, and for example in the FLRW solution.

It would be interesting to understand the effect of a brane wrapping a nonrepresentable cycle on a low energy effective theory in the remaining dimensions, when there are any. For example, the $\mathbb{Z}_3$ anomaly that assures that certain nonrepresentable cycles do not yield K-theory charges may correspond to some interesting anomaly in the low energy effective theory. The low energy physics of D6-branes wrapped on representable cycles that carry torsion K-theory charges has recently been investigated in Ref. [19].

The more interesting question is whether branes can be wrapped on these nonrealizable cycles. For this we need some description of the worldsheet physics. We hope that the worldsheet theory on the product of lens spaces is the IR fixed point of a linear sigma model which is just the tensor product of the linear sigma models on the two lens spaces. This model is somewhat complicated by the strange boundary condition corresponding to the brane which wraps the nonrepresentable cycle.

In the case of the group manifold, the corresponding WZW model is nonunitary. It can be made unitary is we add an H-flux, but in this would go beyond the scope of our results. In particular we do not know how H-flux changes the AHSS differential d_5, although the rational part of the result has been recently provided in Ref. [18]. We hope to use T-duality to find the full expression for d_5 in the twisted case. This would allow one to use a unitary WZW model to determine whether or not a D-brane wrapped on a nonrepresentable cycle can provide a physical boundary condition for fundamental strings. However, once H corrections are included, it may well be that representability will be replaced with a twisted notion of representability, such as the representability of a section of $PU(\mathcal{H})$ or loop group of E_8 bundle over a cycle.

Acknowledgments

We would like to thank Peter Bouwknegt, Alan Carey, Varghese Mathai, Tony Pantev, Eric Sharpe and Bai-Ling Wang for useful discussions. We also thank the organizers of the workshop "Gerbes, Groupoids, and Quantum Field Theory" at the Erwin Schrödinger Institute for the invitation and the stimulating environment. The work of J.E. is partially supported by IISN - Belgium (convention 4.4505.86), by the "Interuniversity Attraction Poles Programme – Belgian Science Policy" and by the European Commission programme MRTN-CT-2004-005104, in which he is associated to V.U. Brussel. The work of H.S. is supported by an ESI Junior Research Fellowship.

References

[1] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, hep-th/9907189.

[2] J.M. Maldacena, G.W. Moore and N. Seiberg, *D-brane instantons and K-theory charges*, *JHEP* **11** (2001) 062 [hep-th/0108100].

[3] R. Minasian and G.W. Moore, *K-theory and Ramond-Ramond charge*, *JHEP* **11** (1997) 002 [hep-th/9710230].

[4] E. Witten, *D-branes and K-theory*, *JHEP* **12** (1998) 019 [hep-th/9810188].

[5] P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted K-theory*, *JHEP* **03** (2000) 007 [hep-th/0002023].

[6] C. Bohr, B. Hanke, and D. Kotschick, *Cycles, submanifolds, and structures on normal bundles*, *Manuscripta Math.* **108** (2002) 483, [math.GT/0011178].

[7] A.L. Carey and B.-L. Wang, *Fusion of symmetric D-branes and verlinde rings*, math-ph/0505040.

[8] P. Hořava, *Type IIA D-branes, K-theory and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1999) 1373 [hep-th/9812135].

[9] A. Sen, *Tachyon condensation on the brane antibrane system*, *JHEP* **08** (1998) 012 [hep-th/9805170].

[10] J. Evslin, *Twisted k-theory as a brst cohomology*, hep-th/0605049.

[11] H. Sati, unpublished.

[12] D.-E. Diaconescu, G.W. Moore and E. Witten, E_8 *gauge theory and a derivation of K-theory from M-theory*, *Adv. Theor. Math. Phys.* **6** (2003) 1031 [hep-th/0005090].

[13] I. Kriz and H. Sati, *M-theory, type-IIA superstrings and elliptic cohomology*, *Adv. Theor. Math. Phys.* **8** (2004) 345 [hep-th/0404013].

[14] R. Thom, *Quelques propriétés globales des variétés différentiables*, *Comment. Math. Helv.* **28** (1954) 17

[15] H. Sati, *Duality-symmetry and the form-fields in M-theory*, *JHEP* **06** (2006) 062 [hep-th/0509046].

[16] P. Bouwknegt, J. Evslin, B. Jurco, V. Mathai and H. Sati, *Flux compactifications of projective spaces and the S-duality puzzle*, *Adv. Theor. Math. Phys.* **10** (2006) 345 [hep-th/0501110].

[17] D. Sullivan, *René Thom's work on geometric homology and bordism*, *Bull. Amer. Math. Soc. (N.S.)* **41** (2004) 341.

[18] M. Atiyah and G. Segal, *Twisted K-theory and cohomology*, [math.KT/0510674]

[19] F. Marchesano, *D6-branes and torsion*, *JHEP* **05** (2006) 019 [hep-th/0603210].